MTH 605: Topology I

Practice Assignment VIII

1. Show that cone CX of a path-connected space X is contractible.

2. Show that the Mobius band deformation retracts onto its core circle, but not to its boundary circle.

3. Let S_g be the closed orientable surface of genus $g \geq 2$. Consider a separating curve S in S_g that separates S_g into subsurfaces S_{g_1} and S_{g_2} as shown in the figure below, and a nonseparating curve C in S_g (as indicated).

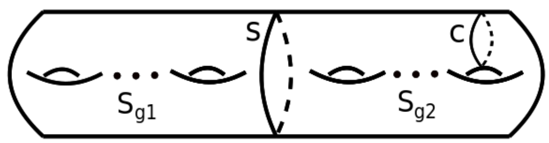

Figure 1: The surface S_g.

   (a) Show that S_g does not retract onto the curve S. [Hint: Does S_{g_2} retract onto S?]

   (b) Show that S_g retracts onto the curve C.

4. Find all covering spaces of the Mobius band and the Klein bottle up to isomorphism. [Hint: See Example 1.42 in page 74 of Hatcher.]

5. Compute the fundamental group of the following spaces.

   (a) The complement of a finite set of points in $\mathbb{R}^n$, for $n \geq 3$.

   (b) The complement of a union of n lines through the origin in $\mathbb{R}^3$.

   (c) The quotient space of S^2 with its north and south poles identified.

   (d) The quotient space obtained from taking two copies of $S^1 \times S^1$ and identifying the circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle in the other torus.

   (e) The complement of a finite set of points in the closed orientable surface S_g of genus g.

6. Suppose that a space Y is obtained from a path-connected space X by attaching n-cells for a fixed $n \geq 3$. Show that the inclusion $X \hookrightarrow Y$ induces an isomorphism of fundamental groups. Use this to show that the complement of a discrete subspace of $\mathbb{R}^n$ is simply connected for $n \geq 3$.

7. The *mapping torus* $M_f(X)$ of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f basepoint-preserving, compute a presentation for $\pi_1(M_f(X))$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$. [Hint: Regard $M_f(X)$ as built from $X \vee S^1$ by attaching cells.]

8. Consider the quotient space X obtained from the solid cube $I^3 = [0,1]^3$ by identifying each square face to its opposite square face via a right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter right twist of the face about its center point.

   (a) Show that X admits a cell complex structure with two 0-cells, four 1-cells, three 2-cells, and one 3-cell.

   (b) Using this structure prove that $\pi_1(X)$ is isomorphic to the quaternion group of order 8.